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Solutions of the Dirac Equation for the Davidson Potential

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Abstract In the present paper we solve the Dirac equation with Davidson potential by Nikiforov-Uvarov method. The Dirac Hamiltonian contains a scalar S and a vector V Davidson potentials. With equal scalar and vector potential, analytical solutions for bound states of the corresponding Dirac equations are found.

Keywords Dirac equation · Anharmonic oscillator potential · Davidson potential · NU method

1 Introduction

The solutions of the Dirac equation play an important role in the relativistic quantum mechanic. One of the application of the relativistic Dirac equation is in the nuclear physics. The Dirac equation with scalar and vector potential is used to describe the dynamic of the particles with spin-$1/2$.

Recently many authors have worked on solving these equations with physical potentials including Morse potential [1, 2], Hulthen potential [3–7], Woods-Saxon potential [8, 9], Posch-Teller potential [10, 11], reflectionless-type potential [12], ring-shaped harmonic oscillator [13], five-parameter exponential potential [14, 15], Rosen-Morse potential [16], generalized symmetrical double-well potential [17], etc.

These methods include the standard method, supersymmetry quantum mechanics [18], the Nikiforov-Uvarov (NU) method [19], etc. The conventional Nikiforov-Uvarov method, which received much interest, has been introduced for solving Schrodinger equation [20–24], Klein-Gordon [25], Dirac and Duffin-Kemmer-Petiau [26] equations. NU method has been used to obtain an explicit exact bound-states solutions for the energy eigenvalues and their corresponding wave functions in terms of orthogonal polynomials for a class of non-central potentials.

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The need for a description of nuclei in which rotational-vibrational interactions dominate has led to search for algebraically solvable potentials. It has been recently shown that one such potential is Davidson potential which has algebraical solutions both for diatomic molecules and for a liquid drop model of the nucleons. The nuclear Davidson potential was introduced by Elliot et al. Davidson proposed such a potential, $V(r)=\chi(r^2+\frac{\varepsilon}{r^2})$ for diatomic molecules, where ε denotes the position of the minimum of the potential and χ is related with the lowest energy of the wave function. the solution is found to be applicable only to well deformed nuclei. The Davidson potential is a scalar function of the nuclear quadrupole moments and expressed in terms of nucleon coordinates. Thus, it is microscopic and rotationally invariant. The rotational-vibrational spectrum of the nuclear system can be generated from a Hamiltonian with Davidson interaction [27]. We find that anharmonic oscillator potential reduces to the Davidson potential in the limiting case of $\beta=0$. The anharmonic oscillator potential is defined as

$$V(r,\theta)=\frac{1}{2}\mu\omega^2r^2+\frac{\hbar^2\alpha}{2\mu r^2}+\frac{\hbar^2\beta\cos^2\theta}{2\mu r^2\sin^2\theta} \tag{1}$$

where μ denote the mass particle ω,α and β are frequency and positive real constants.

Wei and co-workers used the usual algebraic approach to solve the Dirac equation for the anharmonic oscillator potential [28]. In a recent work we want to use the algebraic technique NU to solve Dirac equation with equal scalar and vector anharmonic oscillator potential. This paper is organized as follows.

In Sect. 2, we review the Nikiforov-Uvarov (NU) method briefly. In Sect. 3 we consider the separation of variables for the Dirac equation. Sections 4, 5 devoted to the exact solutions of the radial and angular Dirac equation for the quantum system by the NU method, in Sect. 6 we obtain eigenvalues. Finally, the relevant results are discussed.

2 Nikiforov-Uvarov Method

The second-order differential equations whose solutions are the special functions can be solved by using the NU method. The NU method has been used to solve Schrodinger, Dirac, Klein-Gordon wave equations for certain kind of potentials. In this method the differential equations can be written in the following form

$$\psi(s)''+\frac{\widetilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\widetilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0, \tag{2}$$

where $\sigma(s)$ and $\widetilde{\sigma}(s)$ are polynomials, at most second degree, and $\widetilde{\tau}(s)$ is a first-degree polynomial. To find a particular solution of (2) by separation of variables, we have following transformation

$$\psi(s)=\phi(s)y(s). \tag{3}$$

It reduces (2) to hypergeometric type function

$$\sigma(s)y''+\tau(s)y'+\lambda y=0. \tag{4}$$

The function $\phi(s)$ is defined as a logarithmic derivative

$$\frac{\phi'(s)}{\phi(s)}=\frac{\pi(s)}{\sigma(s)}, \tag{5}$$

$y(s)$ is the hypergeometric type function whose polynomial solutions are given by Rodrigues relation,

$$y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n}[\sigma^n(s)\rho(s)], \tag{6}$$

B_n is the normalizing constant and the weight function ρ must satisfy the following condition

$$(\sigma\rho)' = \tau\rho. \tag{7}$$

The function π and the parameter λ required for this method are defined as

$$\pi(s) = \frac{\sigma' - \widetilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \widetilde{\tau}}{2}\right)^2 - \widetilde{\sigma} + k\sigma}, \tag{8}$$

$$\lambda = k + \pi'. \tag{9}$$

$\pi(s)$ is a polynomial with the parameter s and the determination of k is the essential point in the calculation of $\pi(s)$. For finding the value of k, the expression under the square root must be square of a polynomial, so we have a new eigenvalue equation

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', \tag{10}$$

where

$$\tau(s) = \widetilde{\tau}(s) + 2\pi(s), \tag{11}$$

and it will have a negative derivative. By comparing (9) and (10), we obtain the energy eigenvalues.

3 A Review on the Dirac Equation

The Dirac equation with scalar and vector potentials is

$$[\alpha \cdot p + \beta(\mu + s(r))]\phi(r) = [E - V]\phi(r), \tag{12}$$

$$p = -i\nabla, \qquad \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{13}$$

where σ and I are vector Pauli spin matrix and identity matrix, respectively, P is a momentum, S and V are scalar and vector potentials (here we assume $\hbar = c = \omega = 1$). In Pauli-Dirac representation

$$\phi(r) = \begin{pmatrix} \varphi(r) \\ \chi(r) \end{pmatrix}. \tag{14}$$

Substituting (13) and (14) into (12), we have

$$\sigma \cdot p\chi(r) = [E - V - \mu - s(r)]\varphi(r), \tag{15}$$

$$\sigma \cdot p\varphi(r) = [E - V + \mu - s(r)]\chi(r). \tag{16}$$

With equal scalar and vector potential the above equations become

$$\sigma \cdot p\chi(r) = [E - \mu - 2V]\varphi(r), \tag{17}$$

$$\chi(r) = \frac{\sigma \cdot p}{E+\mu}\varphi(r). \tag{18}$$

By eliminating $\chi(r)$ between these two, we have

$$[p^2 + 2(E+\mu)V(r)]\varphi(r) = [E^2 - \mu^2]\varphi(r). \tag{19}$$

We insert the anharmonic oscillator potential in the above equation. In spherical coordinate, let

$$\varphi(r) = \frac{u(r)}{r} H(\theta)e^{im\phi}, \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{20}$$

By substituting (20) into (19) and using the separation of variables, for $H(\theta)$ and $u(r)$ we have the following equations

$$\frac{d^2H(\theta)}{d\theta^2} + \cot\theta \frac{dH(\theta)}{d\theta} + \left[\ell(\ell+1) - \frac{(m')^2}{\sin^2\theta}\right]H(\theta) = 0, \tag{21}$$

$$\frac{d^2u(r)}{dr^2} - \left[B + a\mu r^2 + \frac{\frac{a\alpha}{\mu} + \lambda}{r^2}\right]u(r) = 0, \tag{22}$$

where λ is the separation constant, a, B, ℓ, m' are defined as

$$a = \mu + E, \tag{23}$$

$$B = \mu^2 - E^2, \tag{24}$$

$$\ell(\ell+1) = \lambda + a\frac{\beta}{\mu}, \tag{25}$$

$$m' = \sqrt{\frac{a\beta}{\mu} + m^2}. \tag{26}$$

4 The Solution of the Angular Equation

By introducing a new variable $x = \cos(\theta)$, (21) becomes

$$(1-x^2)\frac{d^2H(x)}{dx^2} - 2x\frac{dH(x)}{dx} + \left[\ell(\ell+1) - \frac{m'^2}{1-x^2}\right]H(x) = 0. \tag{27}$$

If we apply the NU method, by comparing (27) with (2), the following expressions are obtained

$$\widetilde{\tau} = -2x, \qquad \sigma = 1 - x^2, \qquad \widetilde{\sigma} = -\ell(\ell+1)x^2 + \ell(\ell+1) - (m')^2.$$

Substituting the above expression into (8), we have

$$\pi(x) = \pm\sqrt{[\ell(\ell+1) - k]x^2 - \ell(\ell+1) + (m')^2 + k}. \tag{28}$$

The constant parameter k can be determined from the condition that the expression under the square root must be the square of a polynomial of first degree so it has a double zero, therefore four possible forms of $\pi(s)$

$$\pi(x) = \begin{cases} \pm m'x, & \text{for } k = \ell(\ell+1) + (m')^2, \\ \pm m', & \text{for } k = \ell(\ell+1). \end{cases} \tag{29}$$

The polynomial $\tau = \widetilde{\tau} + 2\pi$, which has a negative derivative.the suitable form is established by $\pi(x) = -m'x$ for $k = \ell(\ell+1) - (m')^2$, so we have

$$\tau = -2(1+m')x, \qquad \tau'(x) = -2(1+m') < 0. \tag{30}$$

According to (9) and (10) the values n can be obtained

$$n = \ell - m', \quad n = 0, 1, 2, \dots. \tag{31}$$

According to (5) and (7)

$$\phi(x) = (1-x^2)^{m'/2}, \tag{32}$$

$$\rho = (1-x^2)^{m'}. \tag{33}$$

Substituting (33) into (6) y_n can be found to be

$$y_n(x) = B_n(1-x^2)^{-m'} \frac{d^n}{dx^n}[(1-x^2)^{n+m'}]. \tag{34}$$

By using $H(x) = \phi(x)y(x)$, the solution of (27) can be written as

$$H_n(x) = N_n(1-x^2)^{-m'/2} \frac{d^{\ell-m'}}{dx^{\ell-m'}}[(x^2-1)^{\ell}], \tag{35}$$

where $H_n(x)$ stands for the associated-Legendre function $P_{\ell}^{m'}(x)$ and N_n is the normalization constant

$$N = \sqrt{\frac{(2n+2m'+1)n!}{2(n+2m')!}}. \tag{36}$$

Thus the wave function of the angle part of the Dirac equation can be written as

$$H(\theta) = \sqrt{\frac{(2n+2m'+1)n!}{2(n+2m')!}} P_{\ell}^{m'}(\cos\theta). \tag{37}$$

5 The Solution of the Radial Equation

Now we return to study (22), with change variable $x = r^2$ we have

$$\frac{d^2u}{dx^2} + \frac{1}{2x}\frac{du}{dx} - \frac{1}{4x^2}[Bx + a\mu x^2 + L(L+1)]u = 0, \tag{38}$$

in which $a\alpha/\mu + \lambda = L(L+1)$.

By comparing (38) with (2) we have

$$\widetilde{\tau}=1, \qquad \sigma=2x, \qquad \widetilde{\sigma}=-a\mu x^2-Bx-L(L+1). \tag{39}$$

Using (8), $\pi(x)$ can be found as

$$\pi(x)=\frac{1}{2}\pm\sqrt{\frac{1}{4}+a\mu x^2+Bx+L(L+1)+2kx}. \tag{40}$$

According to NU method, the expression in the square root must be the polynomial, so

$$k=\frac{-B\pm\sqrt{a\mu(4L(L+1)+1)}}{2}, \tag{41}$$

we can find four possible functions $\pi(x)$ for each k as

$$\pi(x)=\begin{cases}\frac{1}{2}\pm\sqrt{a\mu}\Big(x+\sqrt{\frac{4L(L+1)+1}{4a\mu}}\Big), & \text{for } k=\frac{-B+\sqrt{a\mu(4L(L+1)+1)}}{2},\\ \frac{1}{2}\pm\sqrt{a\mu}\Big(x-\sqrt{\frac{4L(L+1)+1}{4a\mu}}\Big), & \text{for } k=\frac{-B-\sqrt{a\mu(4L(L+1)+1)}}{2}.\end{cases} \tag{42}$$

By considering the negative derivative of τ, the suitable selection is

$$k=\frac{-B-\sqrt{a\mu(4L(L+1)+1)}}{2}, \qquad \pi(x)=\frac{1}{2}\sqrt{a\mu}\left(x-\sqrt{\frac{4L(L+1)+1}{4a\mu}}\right).$$

From (5), (7) and $a\alpha/\mu+\lambda=L(L+1)$ the function $\phi(x)$, $\rho(x)$ can be obtained:

$$\phi(x)=e^{-\frac{\sqrt{a\mu}}{2}x}x^{(L+1)/2}, \tag{43}$$

$$\rho(x)=e^{-\sqrt{a\mu}x}x^{L+\frac{1}{2}}. \tag{44}$$

Then y_N can be written as

$$y_N(x)=B_N e^{\sqrt{a\mu}x}x^{-(L+1/2)}\frac{d^N}{dx^N}\left[x^{N+L+1/2}e^{-\sqrt{a\mu}x}\right]. \tag{45}$$

We use $u(x)=\phi(x)y(x)$, then we have

$$u(x)=C_N e^{\sqrt{a\mu}x/2}x^{-L/2}\frac{d^N}{dx^N}\left[x^{N+L+1/2}e^{-\sqrt{a\mu}x}\right], \tag{46}$$

so

$$u(x)=C_N x^{(L+1)/2}\,e^{-\frac{\sqrt{a\mu}}{2}x}\,L_N^{L+\frac{1}{2}}(\sqrt{a\mu}x), \tag{47}$$

in which $x=r^2$, on the other hand

$$u(r)=C_N r^{L+1}\,e^{-\frac{\sqrt{a\mu}}{2}r^2}\,L_N^{L+\frac{1}{2}}(\sqrt{a\mu}r^2). \tag{48}$$

6 The Eigenvalues

From (42) and $\lambda = k + \pi'$ we have

$$\lambda = \frac{-(\mu^2 - E^2) - \sqrt{a\mu[4L(L+1)+1]}}{2} - \sqrt{a\mu}. \tag{49}$$

Using the relation $\lambda = \lambda_N = -N\tau'(r) - N(N-1)/2\sigma''$, we have

$$-(\mu^2 - E^2) - 2\sqrt{a\mu}(1+2N) = (2L+1)\sqrt{a\mu}, \tag{50}$$

we obtain the energy equation

$$\frac{\mu^2 - E^2}{\sqrt{a\mu}} + 4N + 2L + 3 = 0, \tag{51}$$

in which

$$L = -\frac{1}{2} + \sqrt{\frac{a\alpha}{\mu} + \lambda + \frac{1}{4}}, \tag{52}$$

$$\lambda = \ell(\ell+1) - \frac{a\beta}{\mu} = (m'+n)(m'+n+1) - \frac{a\beta}{\mu}, \tag{53}$$

$$m' = \sqrt{\frac{a\beta}{\mu} + m^2}. \tag{54}$$

7 Conclusion

We interest in the solutions of the Dirac equation with equal scalar and vector Davidson potential, First we consider the general case of anharmonic oscillator potential (for $\beta = 0$ we have Davidson potential), then we apply the NU method to obtain the energies and eigenfunctions for the Dirac equation. We obtain the solution of the angular part as associated-Legendre polynomial and the radial part is a generalized Laguerre polynomial. Our solutions have a good agreement with earlier studies (see Ref. [28]).

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